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REMARKS ON WEIGHTED HARMONIC BERGMAN FUNCTIONS ON HALF-SPACES

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Abstract. On the setting of the upper half-space $\mathbf{H}$ of the Euclidean n-space, we first estimate the size of partial derivatives of any order for functions in weighted harmonic Bergman spaces and then we show that any partial derivative of weighted harmonic Bergman function behaves *nicely* on any proper half-space contained in $\mathbf{H}$. Also, we show that if any partial derivative of a given weighted harmonic Bergman function u vanishes on $\mathbf{H}$, then so does u.

1. Introduction

For a fixed positive integer $n \geq 2$, let $\mathbf{H} = \mathbf{R}^{n-1} \times (0, \infty)$ be the upper half-space. We write point $z \in \mathbf{H}$ as $z = (z', z_n)$ where $z' \in \mathbf{R}^{n-1}$ and $z_n > 0$.

For $\alpha > -1$, $1 \leq p < \infty$, and $\Omega \subset \mathbf{R}^n$, let $b^p_\alpha(\Omega)$ denote *weighted harmonic Bergman space* consisting of all real-valued harmonic functions u on Ω such that

$$||u||_{b^p_\alpha(\Omega)} := \left(\int_\Omega |u(z)|^p \, dV_\alpha(z) \right)^{1/p} < \infty,$$

where $dV_\alpha(z) = \operatorname{dist}(z, \partial\Omega)^\alpha \, dz$, $\operatorname{dist}(z, \partial\Omega)$ denotes the Euclidean distance from z to the boundary of Ω and dz is the Lebesgue measure on $\mathbf{R}^n$. We let $b^p_\alpha = b^p_\alpha(\mathbf{H})$. Then we can check that $dV_\alpha(z) = z_n^\alpha \, dz$ on $\mathbf{H}$.

Harmonic Bergman spaces are not studied as extensively as their holomorphic counterparts and many results on Bergman spaces has been done for bounded domains. [4] and [9], for example, are good references for holomorphic Bergman spaces. $b^p_0(\Omega)$ is studied in [8] and [5] on the setting of upper half-space and bounded smooth domain in $\mathbf{R}^n$, respectively. $b^p_\alpha(B)$ where B is the open unit ball in $\mathbf{R}^n$ is studied in [3]. Weighted harmonic Bergman functions on $\mathbf{H}$ were studied in [6] and [7].

In this paper we first estimate the size of partial derivatives of any order for functions in weighted harmonic Bergman spaces. Next, we show that if $u \in b^p_\alpha$, then $z_n^{\alpha/p} D^{\vec{\beta}} u(z)$ belongs to $L^p(\Omega)$ for any multi-index $\vec{\beta}$, where Ω is a proper half-space contained in $\mathbf{H}$. Finally, we show that if $u \in b^p_\alpha$ and $D^{\vec{\beta}} u$ vanishes on $\mathbf{H}$ for some multi-index $\vec{\beta}$, then u also vanishes on $\mathbf{H}$.

Constants. Throughout the paper we use the same letter C to denote various constants which may change at each occurrence. The constant C may often depend on the dimension n and some other parameters, but it is always independent of particular functions, points or parameters under consideration. For nonnegative

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quantities A and B, we often write $A \lesssim B$ or $B \gtrsim A$ if A is dominated by B times some *inessential* positive constant. Also, we write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2. Main Results

We first estimate the size of partial derivatives of a function $u \in b^p_\alpha$. To do so, we need one lemma called Cauchy's estimates for harmonic functions. (See [1] for details and related facts.)

Lemma 2.1. *Let $\vec{\beta}$ be a multi-index. Then we have*

$$\left|D^{\vec{\beta}} u(z)\right| \lesssim \frac{M}{r^{|\vec{\beta}|}}$$

for all functions u harmonic and bounded by M on $B(z,r)$.

For the rest of this paper, we fix $\alpha > -1$ and $p \in [1, \infty)$.

Proposition 2.2. *Let $z \in \mathbf{H}$ and let $u \in b^p_\alpha$. If $\vec{\beta}$ is a multi-index, then*

$$\left|D^{\vec{\beta}} u(z)\right| \lesssim \frac{1}{z_n^{|\vec{\beta}|+(n+\alpha)/p}} \left(\int_{B(z,z_n/2)} |u(\zeta)|^p \zeta_n^\alpha \, d\zeta \right)^{1/p}.$$

Proof. Fix $z \in \mathbf{H}$ and let $w \in B(z, z_n/5)$. Apply the volume version of the mean-value property to a harmonic function u on $B(w, w_n/5)$ Then we see from Jensen's inequality that

$$\begin{aligned}
\left|u(w)\right|^p &= \left| \frac{1}{V\big(B(w,w_n/5)\big)} \int_{B(w,w_n/5)} u(\zeta)\, d\zeta \right|^p \\
&\le \frac{1}{V\big(B(w,w_n/5)\big)} \int_{B(w,w_n/5)} |u(\zeta)|^p \, d\zeta \\
&\approx \frac{1}{w_n^n} \int_{B(w,w_n/5)} |u(\zeta)|^p \frac{\zeta_n^\alpha}{w_n^\alpha} \, d\zeta, \\
&= \frac{1}{w_n^{n+\alpha}} \int_{B(w,w_n/5)} |u(\zeta)|^p \zeta_n^\alpha \, d\zeta,
\end{aligned}$$

because $\zeta_n \approx w_n$ on $B(w, w_n/5)$.

Note that $4z_n/5 < w_n < 6z_n/5$, because $w \in B(z, z_n/5)$. Thus, $w_n \approx z_n$ and

$$B(w, w_n/5) \subset B(z, z_n/2).$$

Therefore, we have

$$\left|u(w)\right| \lesssim \frac{1}{z_n^{(n+\alpha)/p}} \left(\int_{B(z,z_n/2)} |u(\zeta)|^p \zeta_n^\alpha \, d\zeta \right)^{1/p}.$$

Then we see from Lemma 2.1 applied to a harmonic function u on $B(z, z_n/5)$ that

$$\left|D^{\vec{\beta}} u(z)\right| \lesssim \frac{1}{z_n^{|\vec{\beta}|+(n+\alpha)/p}} \left(\int_{B(z,z_n/2)} |u(\zeta)|^p \zeta_n^\alpha \, d\zeta \right)^{1/p},$$

as desired. Therefore the proof is complete. □

From Proposition 2.2, we know that

$$\left|u(z)\right| \lesssim \frac{\|u\|_{b^p_\alpha}}{z_n^{(n+\alpha)/p}}.$$

Thus, point evaluation is a bounded linear functional on b^p_α and convergence in b^p_α-norm implies the uniform convergence on each compact subset of $\mathbf{H}$. Therefore b^p_α is a Banach space with b^p_α-norm.

Let $\mathbf{H}_\delta = \{z \in \mathbf{H} \mid z_n > \delta\}$ for $\delta > 0$. If $u \in b^p_\alpha$, then we know from the next proposition that $z_n^{\alpha/p} D^{\vec{\beta}} u$ behaves *nicely* on any proper half-space contained in $\mathbf{H}$ for any multi-index $\vec{\beta}$.

Proposition 2.3. *Let $u \in b^p_\alpha$ and let $f(z) = z_n^{\alpha/p} D^{\vec{\beta}} u(z)$ on $\mathbf{H}$. If $\delta > 0$ and $\vec{\beta}$ is any multi-index, then $f \in L^p(\mathbf{H}_\delta)$.*

Proof. Let $z \in \mathbf{H}_\delta$. Then we know from Proposition 2.2 that

$$\left|D^{\vec{\beta}} u(z)\right| \lesssim \frac{1}{z_n^{|\vec{\beta}|+(n+\alpha)/p}} \left(\int_{B(z,z_n/2)} |u(\zeta)|^p \zeta_n^\alpha \, d\zeta \right)^{1/p}.$$

Therefore we have

$$\begin{aligned}
\|f\|^p_{L^p(\mathbf{H}_\delta)} &= \int_{\mathbf{H}_\delta} |D^{\vec{\beta}} u(z)|^p \; z_n^\alpha \, dz \\
&\lesssim \int_{\mathbf{H}_\delta} \int_{B(z,z_n/2)} |u(\zeta)|^p \zeta_n^\alpha \, d\zeta \; \frac{z_n^\alpha}{z_n^{p|\vec{\beta}|+n+\alpha}} \, dz \\
&\leq \frac{1}{\delta^{p|\vec{\beta}|}} \int_{\mathbf{H}_\delta} \int_{B(z,z_n/2)} |u(\zeta)|^p \frac{\zeta_n^\alpha}{z_n^n} \, d\zeta \, dz,
\end{aligned}$$

because $z_n > \delta$. Let

$$\chi_{B(z,z_n/2)}(\zeta) = \begin{cases} 1 & \text{if } \zeta \in B(z, z_n/2) \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if $\zeta \in B(z, z_n/2)$, then $z_n \approx \zeta_n$ and $z \in B(\zeta, \zeta_n)$. Therefore we get from Fubini's theorem that

$$\begin{aligned}
\|f\|^p_{L^p(\mathbf{H}_\delta)} &\lesssim \frac{1}{\delta^{p|\vec{\beta}|}} \int_{\mathbf{H}_\delta} \int_{\mathbf{H}} \left|u(\zeta)\right|^p \chi_{B(z,z_n/2)}(\zeta) \; \frac{\zeta_n^\alpha}{z_n^n} \, d\zeta \, dz, \\
&\lesssim \frac{1}{\delta^{p|\vec{\beta}|}} \int_{\mathbf{H}} \int_{\mathbf{H}_\delta} \left|u(\zeta)\right|^p \chi_{B(\zeta,\zeta_n)}(z) \; \frac{\zeta_n^\alpha}{z_n^n} \, dz \, d\zeta, \\
&\lesssim \frac{1}{\delta^{p|\vec{\beta}|}} \int_{\mathbf{H}} \left|u(\zeta)\right|^p \left(\int_{\mathbf{H}_\delta} \chi_{B(\zeta,\zeta_n)}(z) \, dz \right) \frac{\zeta_n^\alpha}{\zeta_n^n} \, d\zeta, \\
&\lesssim \frac{1}{\delta^{p|\vec{\beta}|}} \int_{\mathbf{H}} |u(\zeta)|^p \zeta_n^\alpha \, d\zeta = \frac{\|u\|^p_{b^p_\alpha}}{\delta^{p|\vec{\beta}|}}.
\end{aligned}$$

Hence

$$\|f\|_{L^p(\mathbf{H}_\delta)} \lesssim \frac{\|u\|_{b^p_\alpha}}{\delta^{|\vec{\beta}|}},$$

and the proof is complete. □

Before we state the next theorem, first note that if $f \in L^p(\mathbf{H}_\delta)$ for some $\delta > 0$ and f is independent of the j^{th} coordinate variable for some $j \in \{1, 2, \ldots, n\}$, then $f \equiv 0$ on $\mathbf{H}_\delta$. We see this easily by Fubini's theorem. Unlike the case of bounded domains, the following result shows that if $u \in b^p_\alpha$ and $D^{\vec{\beta}}u$ vanishes on $\mathbf{H}$ for some multi-index $\vec{\beta}$, then u also vanishes on $\mathbf{H}$.

Theorem 2.4. *Let $u \in b^p_\alpha$ and let $\vec{\beta}$ be a multi-index. Then $D^{\vec{\beta}}u \equiv 0$ if and only if $u \equiv 0$.*

Proof. One direction is clear. Suppose $D^{\vec{\beta}}u \equiv 0$. We show $u \equiv 0$ by induction on $|\vec{\beta}|$. Note that there is nothing to prove if $|\vec{\beta}| = 0$ We assume this has been proved for any multi-indices of order less than or equal to $m \geq 0$.

Let $j \in \{1, 2, \ldots, n\}$ and let $\vec{\beta}$ be a multi-index satisfying $|\vec{\beta}| = m$. Now suppose that

$$D_j D^{\vec{\beta}}u \equiv 0. \tag{2.1}$$

Note that (2.1) implies that $D^{\vec{\beta}}u$ is independent of the j^{th} coordinate variable. Fix $z \in \mathbf{H}$ and we choose $0 < \delta < z_n$. Then we know from Proposition 2.3 that $f \in L^p(\mathbf{H}_\delta)$, where $f(z) = z_n^{\alpha/p} D^{\vec{\beta}}u(z)$.

Suppose that $j = n$. Because $D^{\vec{\beta}}u$ is independent of the n^{th} coordinate variable and $f \in L^p(\mathbf{H}_\delta)$,

$$\|f\|^p_{L^p(\mathbf{H}_\delta)} = \int_{\partial\mathbf{H}} \left|D^{\vec{\beta}}u(z)\right|^p dz' \int_\delta^\infty z_n^\alpha \, dz_n < \infty.$$

Note that

$$\int_\delta^\infty z_n^\alpha \, dz_n = \infty,$$

since $\alpha > -1$. Therefore, we have $D^{\vec{\beta}}u \equiv 0$ on $\mathbf{H}_\delta$. Now, assume that $j = \{1, \ldots, n-1\}$. Then $D_j f = z_n^{\alpha/p} D_j D^{\vec{\beta}}u \equiv 0$. Therefore we know from the remark above that $f \equiv 0$ and so $D^{\vec{\beta}}u \equiv 0$ on $\mathbf{H}_\delta$. In particular, $D^{\vec{\beta}}u(z) = 0$. Because $z \in \mathbf{H}$ is an arbitrary point, $D^{\vec{\beta}}u \equiv 0$ on $\mathbf{H}$. By our induction hypothesis, we get $u \equiv 0$. Therefore the proof is complete.

□

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